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Research Paper

A Bayesian Look at The Rare Event Distribution

Ehtasham Ahmed Zahoor[1*], Syeda Arjumand Javaid[2]

[1]*Captain Hussain Khan Shaheed Government Boys Post Graduate College Rawalakot*

[2]*Government Girls Degree College Rehra Bagh, Azad Jammu & Kashmir*

**Corresponding author's email:* EAZ.ajk@gmail.com

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Abstract

In this article, Bayesian analysis of parameter (θ) of Poisson distribution under simulated data is conducted. Posterior distributions are obtained under two informative (Gamma and Exponential) and two non-informative (Uniform and Jaffrey's) priors. Five loss functions including Square Error Loss Function (SELF), Weighted Square Error Loss Function (WSELF), LINEX Loss Function (LLF), Quasi Quadratic Loss Function (QQLF) and Precautionary Loss Function (PLF) are used to obtain the Bayes estimators and risks associated with them to study the performance and behavior of the Poisson parameter (θ). From this simulation study we found that gamma distribution is suitable prior for Poisson and Quasi Quadratic Loss Function provides efficient results compared to other Loss functions with minimum risks associated with these estimates.

Introduction

The Poisson distribution was first introduced by Simeon Denis Poisson (1781-1840) and published, together with his probability theory, in 1837 in his work research on the probability of judgments in criminal and civil matters. The work focused on certain random variables N that count, among other things, the number of discrete occurrences (sometimes called 'events' or 'arrivals') that take place during a time-interval of given length. A practical application of this distribution was made by Ladislaus Bortkiewicz in 1898 when he was given the task of investigating the number of soldiers in the Prussian army killed accidentally by horse kicks; this experiment introduced the Poisson distribution to the field of reliability engineering. It fits well when the occurrences of the events are rare, so it is also known as probability distribution of rare events.

A discrete random variable X is said to have a Poisson distribution with parameter λ>0, if for k=0,1,2,3,… the probability mass function of X is given by:

$$f(k;\lambda) = \Pr(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \; ;\lambda > 0$$

Where e is the base of the natural logarithm (e=2.71828…) and k! is the factorial of k. The positive real number λ is equal to the expected value of X and also to its variance i.e.

$$\lambda = E(X) = Var(X)$$

Lim et al. (2001) provided a practical simulation-based Bayesian Analysis of parameter-driven models for time series Poisson data with the AR (1) latent process. Fujisaki et al. (2008) considered jump diffusion processes with compound Poisson process whose jump ranges follow the normal or double exponential distributions and also their Bernoulli approximations. Raftery et al. (1986) discussed a Bayesian approach to estimation and hypothesis testing for a Poisson process with a change-point is developed. Although several research papers have appeared on parameter of Poisson distribution but not much attention has been paid on Bayesian analysis of parameter of Poisson distribution. The main aim of this paper is to study the performance and behavior of parameter of Poisson under different loss functions using informative and non-informative priors.

Materials and Methods: Derivations of Posterior distributions, Prior predictive distributions, Bayes estimators and Bayes risk functions using four different prior distributions are presented in this section.

Likelihood Function

In this subsection, the likelihood function of Poisson distribution has been derived.

The probability mass function of Poisson distribution for a random variable X is:

$$f(x,\theta) = \frac{e^{-\theta}\theta^x}{x!} \qquad , x = 0,1,2,3,\ldots$$

Where is the unknown parameter of the distribution with which we are concerned?

The likelihood function is:

$$L(x,\theta) = \frac{e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

Posterior Distributions: Two non-informative and two informative priors are used to derive the posterior distributions and are presented in this section.

Gamma Prior

The p.d.f of gamma prior with hyperparameters a and b is given by:

$$P(\theta) = \frac{b^a}{\Gamma a}\theta^{a-1}e^{-b\theta} \qquad ,0 < \theta < \infty$$

The posterior distribution is:

$$P(\theta|x) = \frac{\beta^\alpha}{\Gamma\alpha}\theta^{\alpha-1}e^{-\beta\theta}$$

Where $\alpha = \sum_{i=1}^{n} x_i + a$ and $\beta = b + n$.

Exponential Prior

The p.d.f of exponential prior with hyperparameter a is given as:

$$P(\theta) = ae^{-a\theta} \qquad ,0 < \theta < \infty$$

The posterior distribution is:

$$P(\theta|x) = \frac{\beta^\alpha}{\Gamma\alpha}\theta^{\alpha-1}e^{-\beta\theta}$$

Where $\alpha = \sum_{i=1}^{n} x_i + 1$ and $\beta = a + n$.

Uniform Prior

The p.d.f of uniform prior is given as:

$$P(\theta) = 1 \qquad ,0 < \theta < 1$$

The posterior distribution is:

$$P(\theta|x) = \frac{\beta^\alpha}{\Gamma\alpha}\theta^{\alpha-1}e^{-\beta\theta}$$

Where $\alpha = \sum_{i=1}^{n} x_i + 1$ and $\beta = n$.

Jeffery's Prior

The p.d.f of Jeffery's prior is given as:

$$P(\theta) \propto \frac{1}{\theta^{\frac{1}{2}}} \qquad ,0 < \theta < \infty$$

The posterior distribution is:

$$P(\theta|x) = \frac{\beta^{\alpha}}{\Gamma\alpha}\theta^{\alpha-1}e^{-\beta\theta}$$

Where $\alpha = \sum_{i=1}^{n} x_i + \frac{1}{2}$ and $\beta = n$.

Prior Predictive Distributions: In this subsection, prior predictive distributions under different informative and non-informative priors have been presented.

Gamma Prior

The prior predictive distribution using gamma prior is given as:

$$P(y) = \frac{b^{a}\Gamma(a+y)}{\Gamma a(b+1)^{a+y}y!}$$

Exponential Prior

The prior predictive distribution using exponential prior is given as:

$$P(y) = \frac{a}{(a+1)^{y+1}}$$

Uniform Prior

The prior predictive distribution using uniform prior is given as:

$$P(y) = \int_{0}^{\infty} 1.\frac{\theta^{y}e^{-\theta}}{y!}d\theta = 1$$

Jeffery's Prior

The prior predictive distribution using Jeffery's prior is given as:

$$P(y) = \frac{\Gamma\left(y+\frac{1}{2}\right)}{y!}$$

Bayes Estimators and Corresponding Risk Functions: This section discusses the derivation of Bayes estimators and corresponding posterior risks under different loss functions. The Bayes estimators are evaluated under Square Error Loss Function (SELF), Weighted Square Error Loss Function (WSELF), LINEX Loss Function (LLF), Quasi Quadratic Loss Function (QQLF) and Precautionary Loss Function

(PLF). The Bayes estimators and corresponding Posterior risks under different loss functions are presented in the following table.

Table 1: Bayes Estimators and corresponding Posterior risks under different loss functions

Loss Function = $L(\theta, \theta^*)$	Bayes Estimator	Posterior Risk
SELF: $(\theta-\theta^*)^2$	$E(\theta\|x)$	$Var(\theta\|x)$
WSELF: $\theta^{-2}(\theta-\theta^*)^2$	$\frac{E(\theta^{-1}\|x)}{E(\theta^{-2}\|x)}$	$1-\left[\frac{\{E(\theta^{-1}\|x)\}^2}{\{E(\theta^{-2}\|x)\}}\right]$
LLF: $e^{c(\theta^*-\theta)}-c(\theta^*-\theta)-1$	$-\frac{1}{c}\log\{E(e^{-c\theta})\}$	$\log\{E(e^{-c\theta})\}+cE_{(\theta\|x)}(\theta)$
QQLF: $\left(e^{-c\theta^*}-e^{-c\theta}\right)^2$	$-\frac{1}{c}\log\{E(e^{-c\theta})\}$	$e^{-2c\theta^*}+E(e^{-2c\theta})-2e^{-c\theta^*}E(e^{-c\theta})$
PLF: $\frac{(\theta^*-\theta)^2}{\theta^*}$	$\left[E_{(\theta\|x)}(\theta^2)\right]^{\frac{1}{2}}$	$2\left[\{E(\theta^2)\}^{\frac{1}{2}}-E(\theta)\right]$

Gamma Prior

The Bayes estimators and corresponding risks using gamma prior under different loss functions are presented in the following table:

Table 2: Bayes Estimators and corresponding risks under Gamma Prior**.**

Loss Function	**Bayes Estimator**	**Bayes Risk**
SELF	$\frac{\sum_{i=1}^{n}x_i+a}{b+n}$	$\frac{\sum_{i=1}^{n}x_i+a}{(b+n)^2}$
WSELF	$\frac{\sum_{i=1}^{n}x_i+a-2}{b+n}$	$\frac{1}{\sum_{i=1}^{n}x_i+a-1}$
LLF	$-\frac{1}{c}ln\left(\frac{b+n}{b+n+c}\right)^{\sum_{i=1}^{n}x_i+a}$	$ln\left(\frac{b+n}{b+n+c}\right)^{\sum_{i=1}^{n}x_i+a}+c\left(\frac{\sum_{i=1}^{n}x_i+a}{b+n}\right)$
QQLF	$-\frac{1}{c}ln\left(\frac{b+n}{b+n+c}\right)^{\sum_{i=1}^{n}x_i+a}$	$\left(\frac{b+n}{b+n+2c}\right)^{\sum_{i=1}^{n}x_i+a}-\left(\frac{b+n}{b+n+c}\right)^{2[\sum_{i=1}^{n}x_i+a]}$

PLF	$\frac{\left[\left(\sum_{i=1}^{n} x_i + a + 1\right)\left(\sum_{i=1}^{n} x_i + a\right)\right]^{\frac{1}{2}}}{b+n}$	$2\left[\frac{\left[\left(\sum_{i=1}^{n} x_i + a + 1\right)\left(\sum_{i=1}^{n} x_i + a\right)\right]^{\frac{1}{2}} - \left(\sum_{i=1}^{n} x_i + a\right)}{b+n}\right]$

Exponential Prior

The Bayes estimators and corresponding risks using exponential prior under different loss functions are presented in the following table:

Table 3: Bayes Estimators and corresponding risks under Exponential Prior.

Loss Function	Bayes Estimator	Bayes Risk
SELF	$\frac{\sum_{i=1}^{n} x_i + 1}{a+n}$	$\frac{\sum_{i=1}^{n} x_i + 1}{(a+n)^2}$
WSELF	$\frac{\sum_{i=1}^{n} x_i - 1}{a+n}$	$\frac{1}{\sum_{i=1}^{n} x_i}$
LLF	$-\frac{1}{c} ln\left(\frac{a+n}{a+n+c}\right)^{\sum_{i=1}^{n} x_i+1}$	$ln\left(\frac{a+n}{a+n+c}\right)^{\sum_{i=1}^{n} x_i+1} + c\left(\frac{\sum_{i=1}^{n} x_i + 1}{a+n}\right)$
QQLF	$-\frac{1}{c} ln\left(\frac{a+n}{a+n+c}\right)^{\sum_{i=1}^{n} x_i+1}$	$\left(\frac{a+n}{a+n+2c}\right)^{\sum_{i=1}^{n} x_i+1} - \left(\frac{a+n}{a+n+c}\right)^{2\left[\sum_{i=1}^{n} x_i+1\right]}$
PLF	$\frac{\left[\left(\sum_{i=1}^{n} x_i + 2\right)\left(\sum_{i=1}^{n} x_i + 1\right)\right]^{\frac{1}{2}}}{a+n}$	$2\left[\frac{\left[\left(\sum_{i=1}^{n} x_i + 2\right)\left(\sum_{i=1}^{n} x_i + 1\right)\right]^{\frac{1}{2}} - \left(\sum_{i=1}^{n} x_i + 1\right)}{a+n}\right]$

Uniform Prior

The Bayes estimators and corresponding risks using uniform prior under different loss functions are presented in the following table:

Table 4: Bayes Estimators and corresponding risks under Uniform Prior.

Loss Function	Bayes Estimator	Bayes Risk
SELF	$\frac{\sum_{i=1}^{n} x_i + 1}{n}$	$\frac{\sum_{i=1}^{n} x_i + 1}{n^2}$
WSELF	$\frac{\sum_{i=1}^{n} x_i - 1}{n}$	$\frac{1}{\sum_{i=1}^{n} x_i}$
LLF	$-\frac{1}{c} ln\left(\frac{n}{n+c}\right)^{\sum_{i=1}^{n} x_i+1}$	$ln\left(\frac{n}{n+c}\right)^{\sum_{i=1}^{n} x_i+1} + c\left(\frac{\sum_{i=1}^{n} x_i + 1}{n}\right)$
QQLF	$-\frac{1}{c} ln\left(\frac{n}{n+c}\right)^{\sum_{i=1}^{n} x_i+1}$	$\left(\frac{n}{n+2c}\right)^{\sum_{i=1}^{n} x_i+1} - \left(\frac{n}{n+c}\right)^{2[\sum_{i=1}^{n} x_i+1]}$
PLF	$\frac{[(\sum_{i=1}^{n} x_i + 2)(\sum_{i=1}^{n} x_i + 1)]^{\frac{1}{2}}}{n}$	$2\left[\frac{[(\sum_{i=1}^{n} x_i + 2)(\sum_{i=1}^{n} x_i + 1)]^{\frac{1}{2}} - (\sum_{i=1}^{n} x_i + 1)}{n}\right]$

Jeffery's Prior

The Bayes estimators and corresponding risks using Jeffery's prior under different loss functions are presented in the following table:

Table 5: Bayes Estimators and corresponding risks under Jeffery's Prior.

Loss Function	Bayes Estimator	Bayes Risk
SELF	$\frac{\sum_{i=1}^{n} x_i + \frac{1}{2}}{n}$	$\frac{\sum_{i=1}^{n} x_i + \frac{1}{2}}{n^2}$
WSELF	$\frac{\sum_{i=1}^{n} x_i - \frac{3}{2}}{n}$	$\frac{1}{\sum_{i=1}^{n} x_i - \frac{1}{2}}$
LLF	$-\frac{1}{c} ln\left(\frac{n}{n+c}\right)^{\sum_{i=1}^{n} x_i+\frac{1}{2}}$	$ln\left(\frac{n}{n+c}\right)^{\sum_{i=1}^{n} x_i+\frac{1}{2}} + c\left(\frac{\sum_{i=1}^{n} x_i + \frac{1}{2}}{n}\right)$
QQLF	$-\frac{1}{c} ln\left(\frac{n}{n+c}\right)^{\sum_{i=1}^{n} x_i+\frac{1}{2}}$	$\left(\frac{n}{n+2c}\right)^{\sum_{i=1}^{n} x_i+\frac{1}{2}} - \left(\frac{n}{n+c}\right)^{2\left[\sum_{i=1}^{n} x_i+\frac{1}{2}\right]}$

PLF	$\frac{\left[\left(\sum_{i=1}^{n} x_i + \frac{3}{2}\right)\left(\sum_{i=1}^{n} x_i + \frac{1}{2}\right)\right]^{\frac{1}{2}}}{n}$	$2\left[\frac{\left[\left(\sum_{i=1}^{n} x_i + \frac{3}{2}\right)\left(\sum_{i=1}^{n} x_i + \frac{1}{2}\right)\right]^{\frac{1}{2}} - \left(\sum_{i=1}^{n} x_i + \frac{1}{2}\right)}{n}\right]$

Results

The Bayes estimates alongside corresponding risks for different loss functions (SELF, LLF, WSELF, QQLF, PLF) under gamma, exponential, uniform and Jeffery's priors using complete data have been presented. The simulation has been carried out for θ=0.5, 1 and 2 with sample sizes 20, 50, 100, 150 and 500. The risks associated with each estimate have also been presented in the tables. The performance of these estimates has been compared in terms of risks associated with each estimate.

Table 6: Simulation using Bayes Estimators and Risks under Gamma Prior for Simulated Data ($\theta = 0.5$)

Sample Size	**Loss Function**				
	SELF	**WSELF**	**LLF**	**QQLF**	**PLF**
20	0.485242 (0.0240993)	0.385912 (0.114020)	0.473577 (0.0116650)	0.473577 (0.00858639)	0.509469 (0.0484549)
50	0.477753 (0.00952932)	0.437861 (0.0435688)	0.479235 (0.00476371)	0.479235 (0.00357987)	0.518255 (0.0197542)
100	0.497839 (0.00497168)	0.477866 (0.0204703)	0.495370 (0.00246941)	0.495370 (0.00181430)	0.508255 (0.00993746)
150	0.505287 (0.00336555)	0.491966 (0.0133580)	0.503612 (0.00167534)	0.503612 (0.00121504)	0.508606 (0.00663886)
500	0.501567 (0.00100286)	0.497568 (0.00400238)	0.501066 (0.000500764)	0.501066 (0.000366863)	0.502566 (0.00199747)

Table 7: Simulation using Bayes Estimators and Risks under Gamma Prior for Simulated Data ($\theta = 1$)

Sample Size	Loss Function				
	SELF	**WSELF**	**LLF**	**QQLF**	**PLF**
20	0.941253 (0.0467469)	0.841924 (0.0557035)	0.918625 (0.0226273)	0.918625 (0.00691014)	0.965766 (0.0490262)
50	1.00643 (0.0200743)	0.966534 (0.0202195)	0.996520 (0.00990564)	0.996520 (0.00265583)	1.01635 (0.0198483)
100	0.995450 (0.00994107)	0.975477 (0.0101338)	0.990512 (0.00493769)	0.990512 (0.00135081)	1.00043 (0.00996159)
150	0.992121 (0.00660819)	0.978800 (0.00675894)	0.988832 (0.00328950)	0.988832 (0.000905429)	0.995446 (0.00664953)
500	1.00129 (0.00200204)	0.997292 (0.00200088)	1.00029 (0.000999688)	1.00029 (0.000269979)	1.00229 (0.00199846)

Table 8: Simulation using Bayes Estimators and Risks under Gamma Prior for Simulated Data ($\theta = 2$)

Sample Size	Loss Function				
	SELF	**WSELF**	**LLF**	**QQLF**	**PLF**
20	1.99725 (0.0991927)	1.89792 (0.0255006)	1.94924 (0.0480130)	1.94924 (0.00191204)	2.02193 (0.0493596)
50	2.01441 (0.0401797)	1.97452 (0.0100007)	1.99458 (0.0198266)	1.99458 (0.00072925)	2.02436 (0.0198970)
100	1.98311 (0.0198043)	1.96313 (0.00506128)	1.97327 (0.00983672)	1.97327 (0.000378802)	1.98809 (0.00997397)
150	2.01188 (0.0134004)	1.99855 (0.00332167)	2.00521 (0.00667062)	2.00521 (0.000241289)	2.01520 (0.00665516)

500	1.99589 (0.00399070)	1.99189 (0.00100279)	1.99390 (0.00199270)	1.99390 (0.000073842)	1.99689 (0.00199896)

Table 9: Simulation using Bayes Estimators and Risks under Exponential Prior for Simulated Data ($\theta = 0.5$)

Sample size	**Loss Function**				
	SELF	**WSELF**	**LLF**	**QQLF**	**PLF**
20	0.537839 (0.0268647)	0.437940 (0.102378)	0.524838 (0.0130012)	0.524838 (0.00864525)	0.562259 (0.0488406)
50	0.529482 (0.0105854)	0.489499 (0.0392390)	0.524259 (0.00522318)	0.524259 (0.00358461)	0.539386 (0.0198067)
100	0.513835 (0.00513731)	0.493839 (0.0198437)	0.511284 (0.00255166)	0.511284 (0.00181600)	0.518810 (0.00994981)
150	0.512321 (0.00341501)	0.498990 (0.0131824)	0.510621 (0.00169996)	0.510621 (0.00121575)	0.515643 (0.00664422)
500	0.503677 (0.00100731)	0.499677 (0.00398647)	0.503174 (0.000502986)	0.503174 (0.000366941)	0.504676 (0.00199794)

Table 10: Simulation using Bayes Estimators and Risks under Exponential Prior for Simulated Data ($\theta = 1$)

Sample size	**Loss Function**				
	SELF	**WSELF**	**LLF**	**QQLF**	**PLF**
20	0.996465 (0.0497728)	0.896566 (0.0527719)	0.972377 (0.024876)	0.972377 (0.00661307)	1.02113 (0.0493386)
50	1.02867 (0.0205651)	0.988690 (0.0198198)	1.01853 (0.0101475)	1.01853 (0.00260400)	1.03862 (0.0198957)

100	1.00656 (0.0100636)	0.986568 (0.0100324)	1.00157 (0.00499851)	1.00157 (0.00133761)	1.01155 (0.00997327)
150	0.999528 (0.00666262)	0.986197 (0.00671369)	0.996212 (0.00331658)	0.996212 (0.000899527)	1.00286 (0.00665469)
500	1.00351 (0.00200695)	0.999515 (0.00199689)	1.00251 (0.00100214)	1.00251 (0.000269441)	1.00451 (0.00199892)

Table 11: Simulation using Bayes Estimators and Risks under Exponential Prior for Simulated Data ($\theta = 2$)

Sample size	**Loss Function**				
	SELF	**WSELF**	**LLF**	**QQLF**	**PLF**
20	2.05852 (0.102822)	1.95862 (0.0248681)	2.00876 (0.0497606)	2.00876 (0.00176152)	2.08334 (0.0496500)
50	2.04947 (0.0409728)	2.00949 (0.00985075)	2.02925 (0.0202174)	2.02925 (0.000694031)	2.05944 (0.0199434)
100	1.99535 (0.0199495)	1.97536 (0.00503586)	1.98544 (0.00990875)	1.98544 (0.000372418)	2.00035 (0.00998548)
150	2.02006 (0.0134653)	2.00673 (0.00331070)	2.01336 (0.00670287)	2.01336 (0.000238539)	2.02339 (0.00666028)
500	1.99834 (0.00399652)	1.99434 (0.00100179)	1.99635 (0.00199560)	1.99635 (0.000073588)	1.99934 (0.00199942)

Table 12: Simulation using Bayes Estimators and Risks under Uniform Prior for Simulated Data ($\theta = 0.5$)

Sample size	Loss Function				
	SELF	**WSELF**	**LLF**	**QQLF**	**PLF**
20	0.538384 (0.0269192)	0.438384 (0.102378)	0.525357 (0.0130271)	0.525357 (0.00865319)	0.562829 (0.0488901)
50	0.529697 (0.0105939)	0.480808 (0.0399355)	0.524470 (0.00522739)	0.524470 (0.00358596)	0.539604 (0.0198147)
100	0.513939 (0.00513939)	0.493939 (0.0198437)	0.511387 (0.00255269)	0.511387 (0.00181635)	0.518915 (0.00995182)
150	0.512391 (0.00341594)	0.499057 (0.0131824)	0.510690 (0.00170042)	0.510690 (0.00121591)	0.515713 (0.00664512)
500	0.503697 (0.00100739)	0.499697 (0.00398647)	0.503194 (0.000503026)	0.503194 (0.000366956)	0.504696 (0.00199802)

Table 13: Simulation using Bayes Estimators and Risks under Uniform Prior for Simulated Data ($\theta = 1$)

Sample size	Loss Function				
	SELF	**WSELF**	**LLF**	**QQLF**	**PLF**
20	0.997475 (0.0498737)	0.897475 (0.0527719)	0.973339 (0.0241356)	0.973339 (0.00661342)	1.02217 (0.0493886)
50	1.02909 (0.0205818)	0.989091 (0.0198198)	1.01894 (0.0101557)	1.01894 (0.00260396)	1.03904 (0.0199038)
100	1.00677 (0.0100677)	0.986768 (0.0100324)	1.00177 (0.00500053)	1.00177 (0.00133761)	1.01176 (0.00997529)
150	0.999663	0.986330	0.996346	0.996346	1.00299

	(0.00666442)	(0.00671369)	(0.00331747)	(0.000899528)	(0.00665559)
500	1.00356 (0.00200711)	0.999556 (0.0019968)	1.00255 (0.00100222)	1.00255 (0.000269441)	1.00456 (0.00199900)

Table 14: Simulation using Bayes Estimators and Risks under Uniform Prior for Simulated Data ($\theta = 2$)

Sample size	**Loss Function**				
	SELF	**WSELF**	**LLF**	**QQLF**	**PLF**
20	2.06061 (0.103030)	1.96061 (0.0248681)	2.01075 (0.0498599)	2.01075 (0.00175809)	2.08546 (0.0497003)
50	2.05030 (0.0410061)	2.01030 (0.00985075)	2.03007 (0.0202337)	2.03007 (0.000693464)	2.06028 (0.0199515)
100	1.99576 (0.0199576)	1.97576 (0.00503586)	1.98584 (0.00991276)	1.98584 (0.000372271)	2.00075 (0.00998750)
150	2.02034 (0.0134689)	2.00700 (0.00331070)	2.01363 (0.00670467)	2.01363 (0.000238474)	2.02367 (0.00666118)
500	1.99842 (0.00399685)	1.99442 (0.00100179)	1.99643 (0.00199576)	1.99643 (0.000073582)	1.99942 (0.00199950)

Table 15: Simulation using Bayes Estimators and Risks under Jaffrey's prior for Simulated Data ($\theta = 0.5$)

Sample size	**Loss Function**				
	SELF	**WSELF**	**LLF**	**QQLF**	**PLF**
20	0.513384 (0.0256692)	0.413384 (0.107902)	0.500962 (0.0124222)	0.500962 (0.00865901)	0.537803 (0.0488385)
50	0.519697	0.479697	0.496566	0.496566	0.511417

	(0.0103939)	(0.0400243)	(0.00494927)	(0.00358906)	(0.0198045)
100	0.508939 (0.00508939)	0.488939 (0.0200425)	0.506412 (0.00252786)	0.506412 (0.00181662)	0.513915 (0.00995135)
150	0.509057 (0.00339371)	0.495724 (0.0132699)	0.507368 (0.00168935)	0.507368 (0.00121604)	0.512380 (0.00664498)
500	0.502697 (0.00100539)	0.498697 (0.00399443)	0.502195 (0.000502028)	0.502195 (0.000366960)	0.503696 (0.00199801)

Table 16: Simulation using Bayes Estimators and Risks under Jaffrey's prior for Simulated Data ($\theta = 1$)

Sample Size	**Loss Function**				
	SELF	**WSELF**	**LLF**	**QQLF**	**PLF**
20	0.972475 (0.0486237)	0.872475 (0.0542020)	0.948944 (0.0235307)	0.948944 (0.00676618)	0.997161 (0.0493733)
50	1.01909 (0.0203818)	0.979091 (0.0200182)	1.00903 (0.0100570)	1.00903 (0.00262998)	1.02904 (0.0199028)
100	1.00177 (0.0100177)	0.981768 (0.0100830)	0.996792 (0.00497569)	0.996792 (0.00134424)	1.00676 (0.00997517)
150	0.996330 (0.00664220)	0.982997 (0.00673630)	0.993024 (0.00330641)	0.993024 (0.000902495)	0.999658 (0.00665555)
500	1.00256 (0.00200511)	0.998556 (0.00199889)	1.00155 (0.00100122)	1.00155 (0.000269711)	1.00356 (0.00199900)

Table 17: Simulation using Bayes Estimators and Risks under Jaffrey's prior for Simulated Data (θ = 2)

Sample size	Loss Function				
	SELF	WSELF	LLF	QQLF	PLF
20	2.03561 (0.101780)	1.93561 (0.0251812)	1.98635 (0.0492550)	1.98635 (0.00182255)	2.06045 (0.0496967)
50	2.04030 (0.0408061)	2.00030 (0.00989951)	2.02017 (0.0201350)	2.02017 (0.000703815)	2.05028 (0.0199512)
100	1.99076 (0.0199076)	1.97076 (0.00504857)	1.98087 (0.00988792)	1.98087 (0.000375042)	1.99575 (0.00998747)
150	2.01700 (0.0134467)	2.00367 (0.00331619)	2.01031 (0.00669361)	2.01031 (0.000239665)	2.02033 (0.00666117)
500	1.99742 (0.00399485)	1.99342 (0.00100229)	1.99543 (0.00199476)	1.99543 (0.000073692)	1.99842 (0.00199950)

Discussion

In this study, the Bayesian analysis of parameter of Poisson distribution has been discussed. Five loss functions including square error loss function (SELF), weighted square error loss function (WSELF), LINEX loss function (LLF), quasi quadratic loss function (QQLF) and precautionary loss function (PLF) have been proposed to estimate the said parameter under the assumption of different (informative and non-informative) priors using complete data. Simulation is done by using the R language.

The simulation study was conducted to analyze the behavior and performance of the estimators of the parameter. From the study, it can be assessed that Bayes estimate of the parameter θ converges to the true value of θ (for which the simulation is done) by increasing the sample size. Under each prior (gamma, exponential, uniform and Jaffrey's) the convergence of the estimates towards the true value of parameter is faster in case of LLF and QQLF. The rate of convergence is random under remaining loss functions. Using WSELF the estimated value of parameter is always less than the estimates obtained under other loss functions. The Bayes estimates obtained under LLF and QQLF are same for each prior. The convergence under SELF, WSELF and PLF is good but random for any value of parameter under each prior. The risk associated with estimates under QQLF is the minimum for each estimate while the maximum risk is associated with estimates under WSELF (for smaller parameter values) and SELF & LLF (for larger values of parameter) for each prior. It has been observed that for larger parameter values risk associated with estimates for each prior under LLF is higher than other loss functions. The patterns of risks are similar almost for each prior and under every loss function. It can also be observed that the risk associated with SELF estimates is approximately double for $= 0.5$, seven times for $\theta = 1$ and fifty four times for $\theta = 2$ than that of QQLF estimates for each prior. The risk associated with WSELF estimates is approximately ten times greater than that of QQLF estimates for $\theta = 0.5$, for $\theta = 1$ it is approximately seven times and for $\theta = 2$ it is approximately thirteen times greater. The risk associated with LLF estimates is approximately equal for $\theta = 0.5$, four times for $\theta = 1$ and twenty seven times greater than that of QQLF estimates for each prior. Similarly, the risk associated with the PLF estimates is approximately five times for $\theta = 0.5$, seven times for $\theta = 01$ and twenty seven times greater than that of QQLF estimates for each prior. It has been observed that under gamma, uniform and Jaffrey's priors for different sample sizes, the parameter is under and overestimated due to simulation. While in case of exponential prior the parameter is overestimated for most of the samples except for the estimates of WSELF under each prior for which the parameter is underestimated for almost all the sample sizes sue to simulation.

Finally, it can be concluded that the QQLF may be preferred for obtaining the Bayes estimates of parameter θ of Poisson distribution, for each prior, as the risks associated with these estimates are minimum. However, the convergence of the estimates towards true value of the parameter θ under PLF estimates is comparatively faster for uniform and Jaffrey's priors, while in case of gamma and exponential priors the convergence is faster for estimates under SELF & LLF. The performance of all the priors is almost similar for each loss function. So, QQLF estimator under each prior provides most efficient results for parameter θ of Poisson distribution. The risks associated with SELF, LLF and PLF estimates are minimum under gamma prior than those of other priors. While the risks associated with estimates of WSELF are minimum under exponential and uniform priors (and are same). Similarly, the risks associated with QQLF are minimum.

So, for Bayesian analysis of parameter of Poisson distribution gamma prior is best prior and QQLF is best loss function and provides most efficient results.

REFERENCES

Lim YB, and Oh MS. 2001. Bayesian analysis of time series Poisson data. Statistics. 28(2): 259-271.

Fujisaki M, and Zhang D. 2009. Bayesian analysis of compound Poisson mixture model and its application to financial data. International Journal of Innovative Computing, Information and Control. 5(1): 109-117.

Raftery AE, and Akman VE. 1986. Bayesian Analysis of a Poisson Process with a Change-Point. Biometrika. 73(1): 85-89.

Forster JJ. Bayesian Inference for Poisson and Multinomial Log-linear Models. Southampton Statistical Sciences Research Institute.

Yarmohammadi M, and Pazira H. 2010. Classical and Bayesian Estimations on the Generalized Exponential Distribution Using Censored Data. International Journal of Math Analysis. 4(29): 1417 – 1431.

Ali S, Aslam M, and Kazmi SMA. 2010. Bayesian analysis of the Laplace model using noninformative priors for complete and censored data under different priors. 3rd International Conference on Statistics. 1-15.

Aslam M, and Javed M. 2011. Bayesian analysis of Rayleigh model assuming single and mixture prior. Proceedings of 8th International Conference on Recent Advances in Statistics. 269-276.

Kazmi SMA, Aslam M, and Ali S. 2011. On the Bayesian Maxwell distribution under different loss functions. Proceedings of 8th International Conference on Recent Advances in Statistics. 25-36.

Saleem M, and Aslam M. 2008. Bayesian analysis of the two-component mixture of Rayleigh distribution with uniform and Jeffery's prior. Journal of Applied Statistical Sciences. 16(4): 105-113.

Saleem M, Aslam M, and Economou P. 2010. On the Bayesian analysis of the mixture of power function distribution using the complete and censored sample. Journal of Applied Statistics. 37(1): 25-40.

Chechile AR. 2001. Bayesian analysis of Gumbel distributed data. Communications in Statistics – Theory and Methods. 30(3): 211-224.

Mousa A. 2002. Bayesian estimation prediction and characterization for the Gumbel model based on records. Statistics. 36(1): 58-69.

Huang WT, and Lin YP 2004. Bayesian sampling plan for exponential distribution based on uniform random censored data. Comp. Stats & Data Analysis. 44: 669-691.

Noortwijk, VMJ, Kalk J H, and Chbab HE 2004. Bayesian estimation of design loads. HERON. 49(2): 189-205.

Liang T. 2004. Empirical Bayes estimation with random right censoring. International Journal of Information and Management Sciences. 15(4): 1-12.